Phase transitions in a network with a range-dependent connection probability

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(Received 9 April 2002; published 20 September 2002)

We consider a one-dimensional network in which the nodes at Euclidean distance l can have long range connections with a probabilty $P(l)\sim l^{-\delta}$ in addition to nearest neighbor connections. This system has been shown to exhibit small-world behavior for $\delta<2$, above which its behavior is like a regular lattice. From the study of the clustering coefficients, we show that there is a transition to a random network at $\delta=1$. The finite size scaling analysis of the clustering coefficients obtained from numerical simulations indicates that a continuous phase transition occurs at this point. Using these results, we find that the two transitions occurring in this network can be detected in any dimension by the behavior of a single quantity, the average bond length. The phase transitions in all dimensions are nontrivial in nature.

DOI: 10.1103/PhysRevE.66.037102 PACS number(s): 89.75.Hc, 05.70.Jk, 64.60.Fr

A network is a collection of nodes that are connected either directly or indirectly by links. There are two extreme examples of networks: the regular and the random network. In a regular network (with infinitely many nodes), the probability that any two arbitrary nodes are connected is vanishingly small while in random networks, this probability remains finite. The two main properties that distinguish these two networks are the chemical distance and the clustering coefficient. The chemical distance is the average shortest distance between any two nodes and is a long distance property. In a network with L nodes, the chemical distance typically behaves as $S_L\sim \ln L$ when it is random, while $S_L\sim L^{1/d}$ in a regular network in d dimensions. The clustering coefficient $\mathcal{C}$ is the average fraction of connected triplets. Since the clustering coefficient measures the local connectivity structure, it is a short range property. Typically $\mathcal{C}$ is high for the regular network and low for the random network.

Recently, another kind of network, the small-world network [1], was proposed which shows random-network-like properties at large scales and regular-network-like properties at small scales. Precisely, the chemical distance S_L behaves as $\ln(L)$ while $\mathcal{C}$ assumes a high value (comparable to a regular network) in this network. Small-world effect can be developed out of a regular lattice having local connections when long range links or connections are allowed to exist even with a very small probability.

The underlying structure of a wide range of networks including social, biological, Internet, and transport networks has been argued to be small-world-like [2]. Additional random long range connections in model systems such as Ising chains or percolation networks also lead to new critical behavior [3–5]. Many of these networks also show scale-free behavior, i.e., if $Q(k)$ is the number of nodes having k connections, then $Q(k)\sim k^{-\gamma}$ in a scale-free network [6].

In the Watts-Strogatz (WS) model [7], the nodes are arranged in a ring. Small-world behavior is observed when the nearest neighbor links are rewired randomly with a probability p. Later it was shown that there is a continuous phase transition occurring at $p\to 0$ [4] from a regular to a small-world phase. For all values of $0<p<1$, the network remains small-world-like with $S_L\sim \ln(L)$ and a high value of $\mathcal{C}$. Only at $p=1$, the network behaves like a random one when $\mathcal{C}$ vanishes. The transitions in the WS model, which is a standard prototype model for small-world behavior, are therefore trivial as the critical points do not separate phases of different critical behavior. In a critical network with nontrivial phase transitions the critical points should have different phases on either side such that the small-world phase emerges as a truly intermediate phase in between the random and regular phases.

In this paper, we show the existence of a network in which such nontrivial transitions can be achieved by tuning an appropriate parameter. In this network [8–10], in addition to the nearest neighbor links, connection to a node at an Euclidean distance l is present with a probability

$$P(l)\sim l^{-\delta}. \tag{1}$$

Such a network can model the behavior of linear polymers in which connections to further neighbors indeed occur with a power law probability. However, in order to simulate the polymer properly, additional constraints have to be considered [10]. Studies on the physical layout of the Internet also strongly suggest that the connections are governed by a power law probability [11].

This network shows small-world effect below $\delta=2$ for the one-dimensional case [8,10] while according to Ref. [9], there is evidence of small-world behavior at $\delta=d$ for the d-dimensional network.

We have shown that there are two transitions occurring in such a network in one dimension: (i) a random to small-world network transition at $\delta=1$; (ii) a small-world to regular network transition at $\delta=2$.

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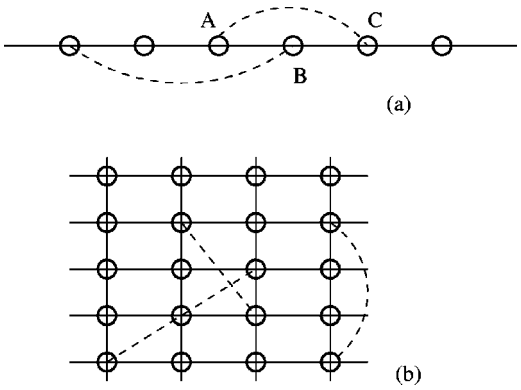

FIG. 1. A section of the network under consideration in (a) one and (b) two dimensions: the solid lines denote the nearest neighbor links between the nodes (circles). The dashed lines denote the long range connections occurring with the probability given in Eq. (1). In (a) the nodes ABC form a triplet cluster.

The second transition is already known to exist [8,10] although the nature of the transition has not been explored in detail. We focus our attention on the first transition which has not been investigated before in either this or any other network to the best of our knowledge. We have been able to locate the critical point and the characteristic critical exponents for this transition.

The other important result of our study is the identification of a single quantity that can detect both the transitions. This analysis can be extended to higher dimensions and the critical points are detected in any dimension d. The phase diagram in the δ-d plane shows that the phase transitions are nontrivial in all dimensions.

First we examine the properties of this network in detail in one dimension. The one-dimensional network consists of L nodes occupying the sites of a one-dimensional lattice of length L such that the nearest neighbor links are of unit length (see Fig. 1). In addition to the nearest neighbor links there are links between nodes at distance l with the probability given by Eq. (1). Note that in one dimension, the Euclidean distance between two nodes coincides with the number of nearest neighbor links separating them. The nearest neighbors links are always present in this network. This corresponds to the situation in many realistic networks such as linear polymers, Ising models, etc. When the probability is normalized one has

$$\sum_{1<l\leqslant L} P(l)=1. \tag{2}$$

The above normalization condition essentially implies that the average number of long distance connections for each node is one. This enables a restriction on the network as the number of long distance bonds is conserved. If the value of δ is made very high, most of the long distance connections will be restricted to the near neighbors and one will effectively get a model with short range connections only. As δ is made smaller, further neighbor bonds will be chosen. For very small values of δ, connections to nodes at all possible distances are made and the network behaves like a random network.

From the above picture and the knowledge of the existence of a nontrivial phase transition from small-world to regular-network-like behavior occurring at $\delta=2$, we expect that all three kinds of behavior (regular, random, and small-world) will be present in this model—or in other words, there will be three regions along the δ axis: $\delta<\delta_c^{(1)}$ where it behaves like a random network, $\delta_c^{(1)}<\delta<\delta_c^{(2)}$ where it is small-world-like and $\delta>\delta_c^{(2)}$ where it is like a regular network. A nonzero value of $\delta_c^{(1)}$ will signify a nontrivial transition. In order to study the transition from the random to the small-world phase it is suffcient to study the clustering coefficients (the chemical distance has similar scaling behavior in both phases). Here we consider clusters which are triplets with three members A, B, and C. The condition that they form a cluster is that if B is connected to A and C, there is also a connection between A and C. Since the nearest neighbors are always present, we classify the possible clusters in three classes.

(1) Clusters with two nearest neighbor (nn) links each of length unity and one next nearest neighbor link of length 2.

(2) Clusters with one nn link, the other two links are of length $l_1>1$ and l_1+1.

(3) Clusters in which there is no nn link; the links have lengths $l_1>1$, $l_2>1$, and l_1+l_2.

Note that the triangular inequality of the link lengths is not valid in one dimension as the distances are always measured along the chain.

Let $\mathcal{C}_i$ be the probability of the occurrence of a cluster belonging to the ith class ($i=1,2,3$). In the continuum limit when l varies continuously, $\mathcal{C}_i$ take the forms

$$\mathcal{C}_1=\frac{2^{-\delta}}{\int P(l)dl}, \tag{3a}$$

$$\mathcal{C}_2=\frac{\int_2^{L-2} P(l_1)P(l_1+1)dl_1}{\left(\int P(l)dl\right)^2}, \tag{3b}$$

$$\mathcal{C}_3=\frac{\int_{l_1,l_2} P(l_1)P(l_2)P(l_1+l_2)dl_1dl_2}{\left(\int P(l)dl\right)^3}. \tag{3c}$$

In the last equation, the integration variables l_1,l_2 satisfy $l_1+l_2<L-2$.

$\mathcal{C}=\Sigma_{i=1}^3\mathcal{C}_i$ is then the clustering coefficient of the system. We find that for $\delta<1$, $\mathcal{C}_i$ vanishes for all i as $L\to\infty$. In particular, to the leading order, $\mathcal{C}_1\sim L^{\delta-1}$ while $\mathcal{C}_2$ and $\mathcal{C}_3$ are $O(L^{-1})$. The vanishing of $\mathcal{C}$ below $\delta=1$ indicates that in this region the network is random. For $\delta>1$, all the three quantities remain finite in the same limit. Hence $\mathcal{C}$ may be interpreted as an order-parameter-like quantity which vanishes at $\delta=1$ where the transition from random to small-world phase takes place.

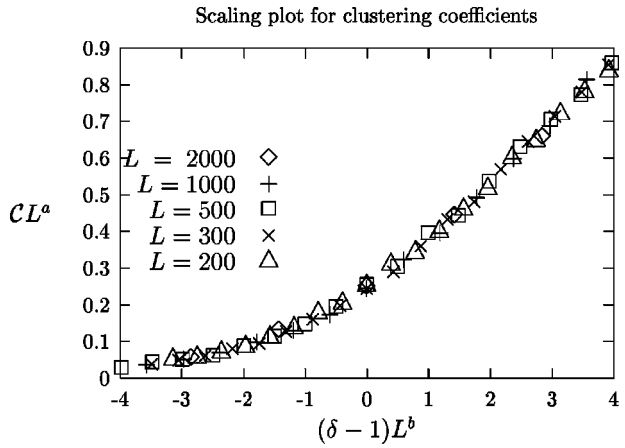

FIG. 2. The data collapse of the scaled clustering coefficients is shown for one-dimensional chains of different lengths L with periodic boundary conditions. Here $a=0.228$ and $b=0.258$

In order to find out the nature of the transition occurring at $\delta=1$ (which we identify as $\delta_c^{(1)}$), we perform numerical simulations and find out the clustering coefficient for chains of different lengths with long range connections existing with a probability given by Eq. (1).

In a thermodynamic system, the finite size scaling for a quantity Φ is given by

$$\Phi(t)=L^{\phi/\nu}f(tL^{1/\nu}), \tag{4}$$

where t is the deviation from the critical point, ϕ is the critical exponent associated with Φ ($\Phi\sim t^{-\phi}$), and ν is the correlation length exponent. We use a finite size scaling form for the clustering coefficients which is analogous to that used in thermodynamic phase transitions,

$$\mathcal{C}=L^{-a}g((\delta_c^{(1)}-\delta)L^b). \tag{5}$$

Using $\delta_c^{(1)}=1$, the value obtained in the continuum case, we find a very good data collapse when $\mathcal{C}L^a$ is plotted against $(\delta-1)L^b$ with the values $a=0.228\pm0.017$ and $b=0.258\pm0.030$ (see Fig. 2). These values are obtained using the Bhattacharjee-Seno method of data collapse [12]. For large values of x, it is expected that $g(x)\sim x^{a/b}$. Note that the value of a/b is the estimate of the exponent β as we have interpreted $\mathcal{C}$ as the "order parameter" [i.e., $\mathcal{C}\sim(\delta-\delta_c^{(1)})^\beta$ for $L\to\infty$], therefore $\beta=0.89\pm0.04$. Also, the finite size scaling form indicates that $\nu=1/b=3.87\pm0.51$.

From the above, we conclude that there is a continuous phase transition occurring at $\delta_c^{(1)}=1$ with characteristic exponents $\nu\sim3.87$ and $\beta=0.89$. Here β describes how the clustering coefficient vanishes as one approaches the random network and the exponent ν is associated with a diverging length scale. Hence in this network the small-world phase appears as an intermediate phase between the random and regular phases and the characterisric behavior of the network can be controlled by tuning the parameter δ.

In general, the clustering coefficient and the chemical distances can detect either of the two phase transitions occurring in a network. This is because they have similar scaling behavior in two of the phases and a different behavior in the third. Interestingly, in the present model, we find a quantity, the average bond length, which shows different scaling behavior in each of these three phases. In the continuum limit, the average bond length $\langle l\rangle$ shows the following scaling behavior:

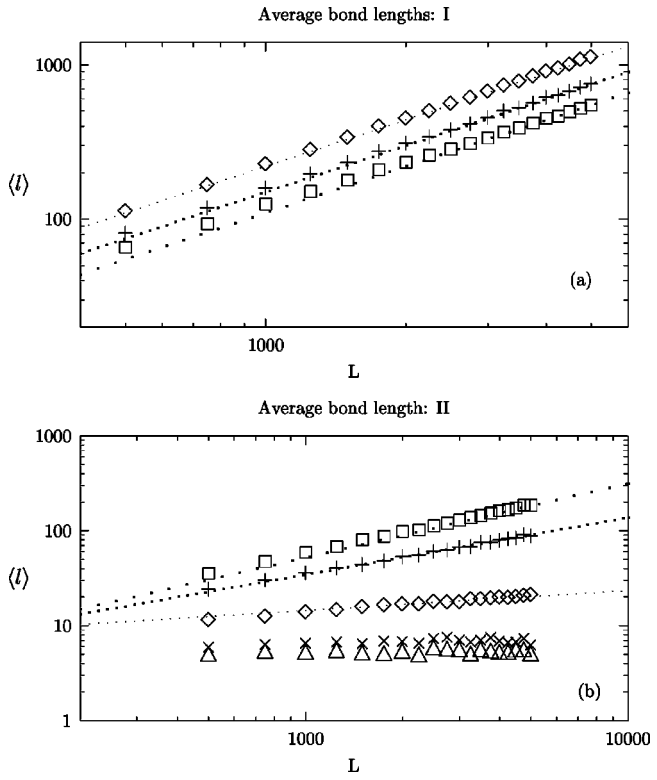

FIG. 3. The scaling behavior of the average bond length $\langle l\rangle$ for one-dimensional chains of different lengths L is shown. In (a) $\langle l\rangle$ for $\delta=0.2$ ($\triangle$), 0.6 (+), and 0.8 ($\square$) along with the best fit lines are shown. The results agree with the behavior $\langle l\rangle\sim L$ for $\delta<1$. In (b) the curves $\langle l\rangle$ are shown for $\delta=1.2$ ($\square$), 1.4 (+), 1.8 ($\diamond$), 2.2 ($\times$), and 2.4 ($\triangle$). The best fit lines for the curves with $1<\delta<2$ are also shown which show agreement with the behavior $\langle l\rangle\sim L^{2-\delta}$ for $1<\delta<2$. For values of $\delta>2$, $\langle l\rangle$ does not depend on L.

$$\langle l\rangle\sim L \quad (\delta<1), \tag{6a}$$

$$\sim L^{2-\delta} \quad (1<\delta<2), \tag{6b}$$

$$\sim O(1) \quad (\delta>2). \tag{6c}$$

We immediately notice that the crossovers occur at $\delta_c^{(1)}$ and $\delta_c^{(2)}$. Hence we find that this is a key quantity since both the transitions can be detected from it. This quantity is also simple to calculate. The numerical values of $\langle l\rangle$ for discrete lattices agree with the above results as shown in Fig. 3.

Although the transition points can be located from the behavior of $\langle l\rangle$, estimating the exponents is not straightforward as it is difficult to cast the behavior of $\langle l\rangle$ in a standard finite size scaling form as in Eq. (5).

The one-dimensional network which we have discussed so far can be generalized to any dimension d where the nodes occupy the sites of a d-dimensional hypercube of length $L^{1/d}$

[see Fig. 1(b)]. Each node is connected to its $2d$ nearest neighbors and long range links to further neighbors occur with a probability given in Eq. (1). (Note that the structure of this network is different from that of a linear polymer embedded in a d-dimensional lattice, although the long range bonds occur with a power law probability in both systems.) For $d>1$, neither $\delta_c^{(1)}$ nor $\delta_c^{(2)}$ is known. In principle, these can be estimated by studying the chemical distance and the clustering coefficients in one dimension but the calculations become much more complicated. The average bond length $\langle l \rangle$, however, can be easily calculated in any general dimension and we find the $\langle l \rangle$ again shows different kinds of behavior in the three regions $0<\delta<d$ (where $\langle l \rangle \sim L$), $d<\delta<d+1$ (where $\langle l \rangle \sim L^{d-\delta+1}$), and $\delta>d+1$ [where $\langle l \rangle \sim O(1)$]. From the results of one dimension we proceed to claim that the transition points in d dimensions are $\delta_c^{(1)}=d$ and $\delta_c^{(2)}=d+1$. This indicates that the small-world region again exists as an intermediate phase between a growing random region and a regular region. The width of the small-world phase is independent of the dimensionality. The phase diagram in the δ-d plane is shown in Fig. 4.

As mentioned earlier, many real networks exhibit scale-free behavior which implies that the degree distribution is a power law. Here we checked that there is no scale-free behavior in any regime in the one-dimensional case. One can expect a scale free behavior when the network shows small-world effect. The absence of scale-free behavior confirms that a small-world network is not necessarily scale-free. However, when distance dependence in the form of Eq. (1) is introduced in a growing network, several interesting features are observed [13].

In summary, our analysis of a model network where the additional long range bonds are present with a probability dependent on the Euclidean distance separating the nodes shows that there is a continuous phase transition occurring at a finite value of the parameter δ where the clustering coefficient behaves like an order parameter. The transition separates a random and a small-world phase. With evidence of a transition from a small-world to regular behavior already existing, we find that this network can be tuned to show regular, random, and small-world behavior for different values of δ. The different behavior occurring on the two sides of the critical points mark the existence of nontrivial phase transitions. This is a feature absent in the familiar WS model of small-world network, where the transitions are reminiscent of the zero temperature phase transitions occurring in the one-dimensional Ising model.

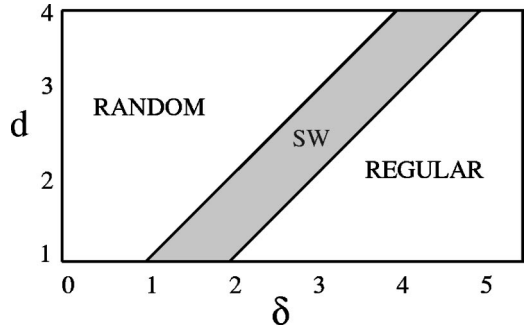

FIG. 4. The phase diagram of the network in the δ-d plane. SW denotes the small-world phase.

Comparing the results obtained from the clustering coefficients and the chemical distances, we find that both the transitions can be detected from the behavior of the average bond lengths. This analysis can be extended to any dimension and the transition points located. We believe that this idea could be useful in general for locating critical points in network whenever the bond length is a meaningful quantity.

P.S. acknowledges DST (India) Grant No. SP/S2/M-11/99.

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